

CLEAR_ONYX

Computational Kernel Briefing

Exact $\mathbb{R}^2$ Operator Algebra, Normal Forms, Phase/Lift Addressing, and Typed Composition

Central result. The kernel generated by the Fibonacci operator Q and the quarter-turn operator C_1 is not merely compact. It spans the full real matrix algebra $M_2(\mathbb{R})$, gives every 2×2 real operator a unique four-coordinate normal form, reduces arbitrary finite operator words exactly, and separates cyclic visible phase from unbounded retained occurrence. The result is a small exact engine with infinite compositional syntax and finite canonical semantics.

AVC

5 August 2026

Technical briefing – print edition

Contents

1 Purpose and scope	2
2 Exact mathematical kernel	2
2.1 Base matrices	2
2.2 Generated quadratic operator	2
2.3 Scalar and register data	3
3 Certificate ledger	3
4 Algebraic completeness and unique normal form	4
5 Two generators and exact word reduction	5
5.1 Minimal generating alphabet	5
5.2 Finite reduction table	6
5.3 Exact coordinate multiplication	6
6 Normalized Clifford core	7
7 The golden seam as audit boundary	7
8 Bounded normalized Fibonacci evolution	8
9 Operational offices	9
10 Phase return and retained occurrence	9
11 Eight channels and twenty-eight relational seams	10
12 Exact computational state and transition contract	11
13 Where free composition lives	12
Implementation reference card	13

1 Purpose and scope

Once the following algebra is organized as a computational engine, it provides substantial freedom to compose while keeping every composition inside a small exact constraint system. Arbitrarily long words in the generating operators can be preserved as provenance, evaluated exactly, and reduced to a unique canonical state. The same kernel can therefore support many lawful rendering, semantic, or physical adapters without allowing those adapters to rewrite the certified mathematics.

Evidence boundary

This note separates three evidence levels.

- **CERT** denotes identities proved directly from the displayed matrices.
- **DERIVED** denotes consequences proved from certified identities together with an explicitly named design choice.
- **DECLARED** denotes architecture selected for the platform, such as the 137-division angular register and the eight-channel carrier. A declared register may be useful and exact as a data type without being forced by the 2×2 kernel itself.

The note certifies an operator algebra and a computational organization. It does not, by itself, certify any particular physical interpretation attached through an adapter.

2 Exact mathematical kernel

2.1 Base matrices

The kernel begins with four explicitly represented matrices acting on $\mathbb{R}^2$:

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\Omega. \quad (1)$$

The matrix Q is symmetric and has characteristic polynomial

$$\chi_Q(\lambda) = \lambda^2 - \lambda - 1. \quad (2)$$

Its eigenvalues are the golden conjugates

$$\Phi = \frac{1 + \sqrt{5}}{2}, \quad \Psi = \frac{1 - \sqrt{5}}{2} = -\Phi^{-1}, \quad (3)$$

with

$$\Phi + \Psi = 1, \quad \Phi\Psi = -1, \quad \Phi - \Psi = \sqrt{5}. \quad (4)$$

2.2 Generated quadratic operator

Define the order-sensitive operator

$$G = Q^T \Omega - \Omega Q. \quad (5)$$

Since $Q^T = Q$, direct multiplication gives

$$G = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}. \quad (6)$$

The normalized operator is

$$H_{\sqrt{5}} = \frac{G}{\sqrt{5}}. \quad (7)$$

2.3 Scalar and register data

The exact scalar layer uses

$$\sqrt{5}, \quad \Phi = \frac{1 + \sqrt{5}}{2}, \quad \tau = 2\pi. \quad (8)$$

The platform architecture additionally uses the following typed registers:

Register	Role	Status
137 angular divisions	Exact cyclic angular address $a \in \mathbb{Z}/137\mathbb{Z}$	DECLARED
8 addressed states	Channel carrier for relational payloads	DECLARED
28 pair relations	Complete unordered pair set among eight channels	DERIVED
4 visible phase positions	Quarter-turn phase class $p \in \mathbb{Z}/4\mathbb{Z}$	CERT
$c \in \mathbb{N}$	Retained completed-lift count	DECLARED
$p \in \{0, 1, 2, 3\}$	Visible phase representative	CERT
$n = 4c + p$	Total occurrence address	DERIVED

3 Certificate ledger

All identities in this section follow by exact matrix multiplication.

Identity	Operational consequence	Status
$Q^2 = Q + I$	Fibonacci lift closes quadratically; every power of Q reduces to the span of I and Q .	CERT
$Q^T \Omega Q = -\Omega$	One Q -lift reverses the oriented area form; two lifts preserve it.	CERT
$Q^T G Q = -G$	One Q -lift reverses the sign of the G -audit form; two lifts preserve it.	CERT
$G^2 = 5I$	The generated audit has exact scale $\sqrt{5}$.	CERT
$H_{\sqrt{5}}^2 = I$	The normalized audit is an involution.	CERT

Identity	Operational consequence	Status
$H_{\sqrt{5}}^T H_{\sqrt{5}} = I$	The normalized involution is Euclidean orthogonal.	CERT
$C_1^2 = -I, C_1^4 = I$	C_1 is an exact quarter-turn with four visible phase positions.	CERT
$\binom{8}{2} = 28$	Eight declared channels have exactly twenty-eight unordered pair relations.	DERIVED
$\Delta_G = 0$ on $y/x = 1/\Phi$ and $y/x = -\Phi$	The quadratic audit vanishes exactly on the two golden eigendirections of Q .	CERT
$\Phi^{-c} Q^c$ remains bounded	Normalized lift can continue for unbounded c without exponential state magnitude.	CERT
$n = 4c + p$	Cyclic phase and retained occurrence have a unique quotient-remainder decomposition.	DERIVED

Two further invariants are useful:

$$\text{tr}(G) = 0, \quad \det(G) = -5, \quad \det(H_{\sqrt{5}}) = -1. \quad (9)$$

Thus $H_{\sqrt{5}}$ is a symmetric orthogonal reflection with eigenvalues $+1$ and -1 .

4 Algebraic completeness and unique normal form

The reason this kernel is useful is stronger than its small size: it supplies the full operator algebra on a two-dimensional real state space.

Proposition 4.1 (Basis theorem). *The ordered set*

$$\mathcal{B} = (I, Q, \Omega, G)$$

is a basis of $M_2(\mathbb{R})$.

Proof. Vectorize each matrix in row-major order and place the vectors as columns:

$$B = \begin{pmatrix} 1 & 1 & 0 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix}. \quad (10)$$

Direct evaluation gives

$$\det(B) = -10 \neq 0. \quad (11)$$

Therefore the four matrices are linearly independent. Since $\dim_{\mathbb{R}} M_2(\mathbb{R}) = 4$, they form a basis. $\square$

Corollary 4.2 (Canonical coordinates). *Every matrix*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R})$$

has one and only one normal form

$$A = x_0 I + x_1 Q + x_2 \Omega + x_3 G.$$

Its coordinates are

$$x_0 = \frac{2a - b - c + 3d}{5}, \quad (12)$$

$$x_1 = \frac{a + 2b + 2c - d}{5}, \quad (13)$$

$$x_2 = \frac{b - c}{2}, \quad (14)$$

$$x_3 = \frac{-2a + b + c + 2d}{10}. \quad (15)$$

This gives every evaluated construction an exact four-coordinate representation. A long operator word and a short operator word are semantically identical precisely when their four normal-form coordinates agree.

$$\text{span}_{\mathbb{R}}\{I, Q, \Omega, G\} = M_2(\mathbb{R})$$

The kernel is a complete four-dimensional real matrix algebra acting on $\mathbb{R}^2$, not a special-purpose animation rule.

5 Two generators and exact word reduction

5.1 Minimal generating alphabet

Because $C_1 = -\Omega$ and $Q^T = Q$,

$$G = Q\Omega - \Omega Q = C_1 Q - Q C_1 = [C_1, Q]. \quad (16)$$

Thus the full algebra is generated by the two-letter alphabet

$$\{Q, C_1\}. \quad (17)$$

The relation

$$Q^2 = Q + I \quad (18)$$

makes Q the Fibonacci growth or lift operator. The relations

$$C_1^2 = -I, \quad C_1^4 = I \quad (19)$$

make C_1 the quarter-turn phase operator. Their commutator

$$G = C_1 Q - Q C_1 \quad (20)$$

measures the exact difference between applying turn then growth and applying growth then turn.

$G = \text{order-sensitive composition audit.}$

5.2 Finite reduction table

Let

$$w = g_1 g_2 \cdots g_m, \quad g_j \in \{Q, C_1\} \quad (21)$$

be any finite operator word. Its evaluated matrix reduces uniquely to

$$\text{NF}(w) = x_0 I + x_1 Q + x_2 \Omega + x_3 G. \quad (22)$$

All products reduce through the finite table

$$Q^2 = Q + I, \quad \Omega^2 = -I, \quad G^2 = 5I, \quad (23)$$

$$Q\Omega = \frac{\Omega + G}{2}, \quad \Omega Q = \frac{\Omega - G}{2}, \quad (24)$$

$$QG = \frac{G + 5\Omega}{2}, \quad GQ = \frac{G - 5\Omega}{2}, \quad (25)$$

$$\Omega G = 2Q - I, \quad G\Omega = I - 2Q. \quad (26)$$

Therefore the engine has

$$\boxed{\text{infinite compositional syntax} + \text{finite exact semantics.}} \quad (27)$$

The unreduced word can be retained as provenance while the four coefficients provide exact comparison, caching, testing, rendering input, and error localization.

5.3 Exact coordinate multiplication

If

$$A = x_0 I + x_1 Q + x_2 \Omega + x_3 G, \quad B = y_0 I + y_1 Q + y_2 \Omega + y_3 G, \quad (28)$$

then $AB = z_0 I + z_1 Q + z_2 \Omega + z_3 G$, where

$$z_0 = x_0 y_0 + x_1 y_1 - x_2 y_2 - x_2 y_3 + x_3 y_2 + 5x_3 y_3, \quad (29)$$

$$z_1 = x_0 y_1 + x_1 y_0 + x_1 y_1 + 2x_2 y_3 - 2x_3 y_2, \quad (30)$$

$$z_2 = x_0 y_2 + \frac{x_1 y_2}{2} + \frac{5x_1 y_3}{2} + x_2 y_0 + \frac{x_2 y_1}{2} - \frac{5x_3 y_1}{2}, \quad (31)$$

$$z_3 = x_0 y_3 + \frac{x_1 y_2}{2} + \frac{x_1 y_3}{2} - \frac{x_2 y_1}{2} + x_3 y_0 + \frac{x_3 y_1}{2}. \quad (32)$$

This bilinear law is a direct implementation interface: multiplication can occur entirely in normal-form coordinates without reconstructing an intermediate symbolic expression.

6 Normalized Clifford core

Set

$$H_{\sqrt{5}} = \frac{G}{\sqrt{5}}. \quad (33)$$

Then

$$H_{\sqrt{5}}^2 = I, \quad \Omega^2 = -I, \quad H_{\sqrt{5}}\Omega = -\Omega H_{\sqrt{5}}. \quad (34)$$

These are the defining relations of the real Clifford algebra of one positive and one negative generator:

$$\text{Cl}_{1,1}(\mathbb{R}) \cong M_2(\mathbb{R}). \quad (35)$$

A normalized operational basis is therefore

$$\{I, H_{\sqrt{5}}, \Omega, \Omega H_{\sqrt{5}}\}. \quad (36)$$

The original lift operator is recoverable from the normalized core. Since $\Omega G = 2Q - I$,

$$Q = \frac{I + \Omega G}{2} = \frac{I + \sqrt{5} \Omega H_{\sqrt{5}}}{2} = \frac{I - \sqrt{5} C_1 H_{\sqrt{5}}}{2}. \quad (37)$$

Thus growth, quarter-turn phase, reflection, and the mixed direction are not separate modules. They are coordinated offices in one exact algebra.

7 The golden seam as audit boundary

For $v = (x, y)^T$, define the quadratic audit

$$\Delta_G(v) = v^T G v. \quad (38)$$

Direct substitution gives

$$\Delta_G(x, y) = -2x^2 + 2xy + 2y^2. \quad (39)$$

Using $\Phi - \Phi^{-1} = 1$, this factors exactly as

$$\Delta_G(x, y) = 2 \left(y - \frac{x}{\Phi} \right) (y + \Phi x). \quad (40)$$

Hence the null directions are

$$\frac{y}{x} = \frac{1}{\Phi}, \quad \frac{y}{x} = -\Phi. \quad (41)$$

These are precisely the eigendirections of Q :

$$v_\Phi = (\Phi, 1)^T, \quad v_\Psi = (\Psi, 1)^T, \quad (42)$$

with

$$Qv_\Phi = \Phi v_\Phi, \quad Qv_\Psi = \Psi v_\Psi. \quad (43)$$

The normalized involution exchanges the two golden directions projectively:

$$H_{\sqrt{5}}v_\Phi = \Phi v_\Psi, \quad H_{\sqrt{5}}v_\Psi = \Phi^{-1}v_\Phi. \quad (44)$$

Therefore

$$\boxed{H_{\sqrt{5}} = \text{the exact involution exchanging the two golden growth directions.}} \quad (45)$$

The identities $Q^T G Q = -G$ and $(Q^2)^T G Q^2 = G$ add a useful parity interpretation: a single lift crosses the sign of the quadratic audit, while a completed pair of lifts preserves it.

8 Bounded normalized Fibonacci evolution

The quadratic relation for Q yields the exact Fibonacci power law

$$Q^c = F_c Q + F_{c-1} I, \quad c \geq 1, \quad (46)$$

where $F_0 = 0$, $F_1 = 1$, and $F_{c+1} = F_c + F_{c-1}$.

Normalize by the dominant eigenvalue:

$$\widehat{Q}_c = \Phi^{-c} Q^c. \quad (47)$$

The spectral projectors are

$$P_\Phi = \frac{Q - \Psi I}{\sqrt{5}}, \quad P_\Psi = \frac{\Phi I - Q}{\sqrt{5}}. \quad (48)$$

They satisfy

$$P_\Phi + P_\Psi = I, \quad P_\Phi P_\Psi = 0, \quad Q = \Phi P_\Phi + \Psi P_\Psi. \quad (49)$$

Consequently,

$$\boxed{\Phi^{-c} Q^c = P_\Phi + \left(-\Phi^{-2}\right)^c P_\Psi.} \quad (50)$$

Because $|\Phi^{-2}| < 1$, the normalized sequence is bounded and converges exponentially to P_Φ .

Computationally, this supplies:

- an unbounded lift count c ;
- bounded normalized operator magnitude;
- an exact retained coefficient for the conjugate mode;
- a stable asymptotic state;
- exact discrete kernel states that do not need to be inferred from interpolation between unrelated rendered endpoints.

The occurrence count can grow without bound while the normalized state remains controlled.

9 Operational offices

The kernel becomes an instrument when each object has a single explicit computational office.

Object	Operational office	Status
I	Identity retention: preserve the current state.	CERT
$C_1 = -\Omega$	Positive quarter-turn phase: open the orthogonal direction.	CERT
Ω	Oriented area form and signed rotational carrier.	CERT
Q	Fibonacci lift: generate the next exact growth state.	CERT
$G = [C_1, Q]$	Detect the difference made by operator order.	CERT
$H_{\sqrt{5}} = G/\sqrt{5}$	Normalize the order difference into an exact orthogonal involution.	CERT
$\Delta_G(v)$	Classify a direction relative to the two golden null seams.	CERT
Φ^{-c}	Normalize accumulated lift while preserving its exact spectral form.	CERT
$p \in \mathbb{Z}/4\mathbb{Z}$	Visible phase position.	CERT
$c \in \mathbb{N}$	Retained count of completed four-phase lifts.	DECLARED
$n = 4c + p$	Unambiguous total occurrence address.	DERIVED
$a \in \mathbb{Z}/137\mathbb{Z}$	Exact angular scope address.	DECLARED
Eight states	Addressed channel carrier.	DECLARED
Twenty-eight pairs	Complete relational seam set.	DERIVED
Append-only R_n	Preserve occurrence even when visible phase returns.	DECLARED

The phrase “every +1” has a literal implementation meaning:

$$\boxed{+1 = \text{apply one typed transition and retain its occurrence.}} \quad (51)$$

It does not mean merely draw one more visual frame.

10 Phase return and retained occurrence

Let $n \in \mathbb{N}$. Euclidean division by four gives the unique decomposition

$$n = 4c + p, \quad 0 \leq p < 4, \quad (52)$$

with

$$p = n \bmod 4, \quad c = \left\lfloor \frac{n}{4} \right\rfloor. \quad (53)$$

Thus visible phase and retained history are separated:

$$p_{n+4} = p_n, \quad c_{n+4} = c_n + 1. \quad (54)$$

The exact computational statement is

$$\boxed{\text{phase closes; occurrence continues.}} \quad (55)$$

The angular register is separately typed as

$$a \in \mathbb{Z}/137\mathbb{Z}. \quad (56)$$

Since $\gcd(4, 137) = 1$, the Chinese remainder theorem gives

$$\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/137\mathbb{Z} \cong \mathbb{Z}/548\mathbb{Z}. \quad (57)$$

The finite 548-address register joins visible phase and angular address exactly. The unbounded count c remains outside it. The system may therefore revisit a finite visible address without revisiting the same occurrence.

For exact angular storage, retain the integer address and interpret

$$\theta_a = \frac{a}{137} \tau \quad (58)$$

symbolically. Floating-point sine, cosine, color conversion, and pixels belong at the rendering boundary rather than in the exact address layer.

11 Eight channels and twenty-eight relational seams

Let the declared channel carrier be

$$A_8 \cong \mathbb{R}^8. \quad (59)$$

Its complete pairwise relation space is the second exterior power

$$\Lambda^2(A_8^*), \quad (60)$$

with dimension

$$\dim \Lambda^2(A_8^*) = \binom{8}{2} = 28. \quad (61)$$

The eight channels are a design choice. Once they are chosen, the twenty-eight unordered pair relations are forced.

If each channel carries a phase, a common phase is relationally invisible. The relative-phase carrier is

$$U(1)^8/U(1)_{\text{diag}} \cong U(1)^7. \quad (62)$$

The architecture therefore keeps distinct:

- eight addressed channel states;
- seven independent relative-phase degrees of freedom;
- twenty-eight complete pairwise reads;
- four visible phase positions;
- 137 angular addresses;
- one unbounded retained lift count.

These axes are typed separately rather than collapsed into one opaque index.

12 Exact computational state and transition contract

A minimal state record may be written

$$X_n = (A_n, p_n, c_n, a_n, u_n, R_n), \quad (63)$$

where

$$A_n = x_{0,n}I + x_{1,n}Q + x_{2,n}\Omega + x_{3,n}G \quad (64)$$

is the exact operator normal form,

$$p_n \in \mathbb{Z}/4\mathbb{Z}, \quad c_n \in \mathbb{N}, \quad a_n \in \mathbb{Z}/137\mathbb{Z}, \quad u_n \in K^8, \quad (65)$$

and K is the scalar or phase domain declared by the active adapter. The occurrence ledger is append-only:

$$R_{n+1} = R_n \frown o_{n+1}. \quad (66)$$

Every transition should retain at least:

1. the generator applied;
2. the unreduced operator word;
3. the exact reduced normal form;
4. the input and output addresses;
5. the audit result;
6. any rendered approximation produced at the boundary.

This record allows the platform to identify where a fracture entered rather than merely displaying a wrong final result.

A typed transition may be represented schematically as

$$T_g : X_n \mapsto X_{n+1}, \quad g \in \{Q, C_1\}, \quad (67)$$

subject to the invariants

$$A_{n+1} = \text{NF}(A_n g), \quad p_{n+1} = p_n + 1 \pmod{4}, \quad R_{n+1} = R_n \frown o_{n+1}, \quad (68)$$

with adapter-specific updates to a_n and u_n declared separately.

13 Where free composition lives

The certified identities remain fixed. Compositional freedom lies in choosing:

- which legal generator word to execute;
- which normal-form coordinate or invariant to observe;
- which channel or seam to read;
- which angular address to expose;
- which semantic or physical adapter interprets the state;
- which visual grammar renders it;
- which typed transition policy selects the next legal word.

A color loom, symmetry viewer, phase clock, grammatical construction, or physical simulation may therefore be implemented as a distinct adapter around the same core. Each adapter must declare what its coordinates mean and must not promote visual resemblance into an algebraic or physical certificate.

The architecture can be summarized as

$$\begin{array}{c} \text{operator words} \xrightarrow{\text{Eval}} M_2(\mathbb{Q}(\sqrt{5})) \xrightarrow{\text{NF}} (x_0, x_1, x_2, x_3) \\ \xrightarrow{\text{typed address}} (p, c, a, u, \text{seam}, R) \xrightarrow{\text{adapter}} \text{visible or physical realization.} \end{array} \quad (69)$$

For words in the unnormalized generators Q and C_1 , evaluation already lies in $M_2(\mathbb{Z})$ and normal-form coordinates lie in $\mathbb{Q}$. The extension $\mathbb{Q}(\sqrt{5})$ enters when the normalized involution $H_{\sqrt{5}}$ or golden spectral projectors are used.

The core provides the five structural properties required of a reusable generative engine:

exact syntax + canonical reduction + bounded normalized evolution + unbounded memory + typed free adapters.

The kernel is not the finished composition. It is the small exact machine that permits many compositions without allowing any one of them to quietly rewrite the rules.

Implementation reference card

Generators

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Omega = -C_1.$$

Generated operators

$$G = [C_1, Q] = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}, \quad H_{\sqrt{5}} = G/\sqrt{5}.$$

Core relations

$$\begin{aligned} Q^2 &= Q + I, & C_1^2 &= -I, & G^2 &= 5I, & H_{\sqrt{5}}^2 &= I, \\ Q^T \Omega Q &= -\Omega, & Q^T G Q &= -G, & H_{\sqrt{5}} \Omega &= -\Omega H_{\sqrt{5}}. \end{aligned}$$

Normal form

$$A = x_0 I + x_1 Q + x_2 \Omega + x_3 G.$$

Golden audit

$$\Delta_G(x, y) = 2 \left(y - \frac{x}{\Phi} \right) (y + \Phi x), \quad \Phi = \frac{1 + \sqrt{5}}{2}.$$

Normalized lift

$$\Phi^{-c} Q^c = P_\Phi + (-\Phi^{-2})^c P_\Psi.$$

Address

$$n = 4c + p, \quad p = n \bmod 4, \quad c = \lfloor n/4 \rfloor, \quad a \in \mathbb{Z}/137\mathbb{Z}.$$

Relational carrier

$$\dim \Lambda^2((\mathbb{R}^8)^*) = \binom{8}{2} = 28, \quad U(1)^8 / U(1)_{\text{diag}} \cong U(1)^7.$$

State

$$X_n = (A_n, p_n, c_n, a_n, u_n, R_n), \quad R_{n+1} = R_n \frown o_{n+1}.$$