ONE CARRIER, THREE GEOMETRIES

Symplectic Orientation, Lorentzian Null Structure, and Positive Phase from Two Integer Matrices

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Abstract

We construct an exact two-dimensional real carrier from the integer matrices

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The commutator $G = [Q, \Omega]$ is symmetric of signature $(1, 1)$, and its two projective null directions are exactly the golden eigendirections of Q. After normalization, G and $2Q - I$ define two Euclidean reflections whose ordered products are the quarter-turn Ω and its inverse. A further composite J satisfies $J^2 = -I$, while the separately derived form $B = -\Omega J$ is positive definite with determinant one and spectrum $\{\varphi^{-1}, \varphi\}$. In the golden basis, J is the standard quarter-turn and $B = \sqrt{5}I$. Projectively, the transport and mirror generate D_∞, while the visible phase factors through D_4. Finally, the same phase generator is used in an explicit Lorentzian $SO(2)$ gauge model carrying nonzero source-free Maxwell curvature, arbitrarily prescribed loop holonomy, and unequal proper time between endpoint-sharing timelike paths. Every statement is exact. No identification is asserted among the alternating form, the Lorentzian form, the positive metric, spacetime curvature, gauge holonomy, or metric duration.

Keywords: alternating form; Lorentzian plane; golden null rays; compatible complex structure; positive metric; infinite dihedral group; $SO(2)$ holonomy; source-free Maxwell field; proper time.

Jurisdiction: real linear algebra, bilinear forms, projective geometry, finite and infinite dihedral groups, Abelian connections, Lorentzian geometry, and source-free Maxwell theory. The three carrier geometries and the spacetime gauge witness are separately typed throughout.

1 Statement and scope

The purpose of the construction is not to identify several familiar geometries. It is to determine how much exact structure can be generated by one minimal carrier while the resulting offices remain distinct.

The carrier begins with an orientation form and one reciprocal transport. The commutator of their orderings produces an indefinite symmetric form. Two normalized operators then become ordinary Euclidean reflections. Their order produces phase, and a second composite produces a positive metric compatible with a complex structure. The same orientation generator can separately be used as the Lie-algebra value of an $SO(2)$ connection on a Lorentzian spacetime. The standard background on symplectic forms, reflection groups, connections, and Lorentzian proper time is classical [3, 1, 2, 4, 5]. The present note isolates the smallest common carrier from the broader constructions of Refs. [6, 7].

Theorem 1.1 (Exact carrier and Lorentzian witness)**.** *There exists a tuple*

$$\mathfrak{A} = \left(V, Q, \Omega, \omega, G, g_G, H, S, J, B, b, \varphi, \psi, D_\infty, D_4, M, g_L, E, \mathcal{A}_a, F, \gamma_0, \gamma_1\right)$$

with the following properties.

(i) *Q and Ω generate all of $M_2(\mathbb{R})$ through the exact basis $\{I, Q, \Omega, G\}$, where $G = [Q, \Omega]$.*

(ii) *ω is nondegenerate alternating, g_G has signature $(1, 1)$, and one Q-step reverses both forms while two steps restore both forms without implying state return.*

(iii) *the two null rays of g_G are the eigendirections of Q with slopes $\varphi = (1+\sqrt{5})/2$ and $\psi = (1-\sqrt{5})/2$.*

(iv) *$H = G/\sqrt{5}$ and $S = (2Q - I)/\sqrt{5}$ are Euclidean reflections with $SH = \Omega$ and $HS = \Omega^{-1}$.*

(v) *$J = QH$ satisfies $J^2 = -I$, and $B = -\Omega J$ is positive definite, J-invariant, unimodular, and has spectrum $\{\varphi^{-1}, \varphi\}$.*

(vi) *the projective transport $q = [Q]$ and mirror $h = [H]$ generate D_∞, and the phase representation $q \mapsto \Omega$, $h \mapsto H$ is a surjection $D_\infty \twoheadrightarrow D_4$.*

(vii) *on an explicit Lorentzian four-manifold, the same Ω is the Lie-algebra generator of an $SO(2)$ connection whose nonzero source-free Maxwell curvature is independent of a parameter a, while the holonomy around a fixed loop ranges over all of $SO(2)$.*

(viii) *two explicit timelike paths with common endpoints have unequal proper time, independently of the holonomy parameter.*

No equality or functional dependence is asserted among G, B, the Lorentzian spacetime metric g_L, the Maxwell curvature F, loop holonomy, and proper-time difference.

The proof occupies Sections 2–5. Section 6 records the strict boundary.

2 The generated finite kernel

Let $V = \mathbb{R}^2$ with its standard Euclidean inner product. Define

$$Q := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Omega := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \omega(u, v) := u^{\mathsf{T}} \Omega v.$$

Then $\Omega^{\mathsf{T}} = -\Omega$, $\Omega^2 = -I$, and ω is a nondegenerate alternating form. After Euclidean index raising, the same matrix is a clockwise quarter-turn. Its one-parameter phase is

$$U_\Omega(\theta) = e^{\theta\Omega} = I\cos\theta + \Omega\sin\theta \in \mathrm{SO}(2).$$

Define the generated symmetric operator

$$G := [Q, \Omega] = Q\Omega - \Omega Q = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad g_G(u, v) := u^{\mathsf{T}} G v.$$

The symmetry of G is derived from operator order; it is not separately postulated.

Proposition 2.1 (Complete local algebra)**.** *The set $\{I, Q, \Omega, G\}$ is a basis of $M_2(\mathbb{R})$. Moreover,*

$$\{Q, \Omega\} = \Omega, \qquad [Q, \Omega] = G, \qquad \{Q, G\} = G, \qquad [Q, G] = 5\Omega,$$

where $\{A, B\} = AB + BA$. Consequently

$$\mathrm{ad}_Q^2(\Omega) = 5\Omega.$$

Proof. Suppose $aI + bQ + c\Omega + dG = 0$. Comparison of the off-diagonal entries gives $c = 0$ and $b + d = 0$; the $(2,2)$ entry gives $a + 2d = 0$; the $(1,1)$ entry then gives $-5d = 0$. Hence $a = b = c = d = 0$. Since $\dim M_2(\mathbb{R}) = 4$, the four matrices form a basis. The product relations follow by direct multiplication. □

The reciprocal law

$$Q^2 = Q + I, \qquad \det Q = -1, \qquad \mathrm{tr}\, Q = 1$$

will control both the null geometry and the projective transport.

Proposition 2.2 (Dual-form reversal)**.** *One transport step reverses both governing forms:*

$$Q^{\mathsf{T}} \Omega Q = -\Omega, \qquad Q^{\mathsf{T}} G Q = -G.$$

Consequently

$$(Q^2)^{\mathsf{T}} \Omega Q^2 = \Omega, \qquad (Q^2)^{\mathsf{T}} G Q^2 = G.$$

The second pair is form return, not state return: generically $Q^2 v \neq v$.

Proof. The alternating identity is the two-dimensional formula $A^{\mathsf{T}}\Omega A = (\det A)\Omega$ applied to $A = Q$. The symmetric identity follows by direct multiplication. Applying either reversing identity twice gives the two-step return. Since $Q^2 \neq I$, state return does not follow. □

3 Three geometries on one carrier

3.1 Lorentzian null structure

The generated form satisfies

$$G^{\mathsf{T}} = G, \qquad G^2 = 5I, \qquad \det G = -5.$$

Its eigenvalues are $\pm\sqrt{5}$, hence g_G has signature $(1,1)$. For $v = (x,y)^{\mathsf{T}}$,

$$g_G(v,v) = -2(x^2 - xy - y^2).$$

If $y \neq 0$, the null condition is

$$\left(\frac{x}{y}\right)^2 - \frac{x}{y} - 1 = 0.$$

Thus the two projective null rays have slopes

$$\varphi = \frac{1+\sqrt{5}}{2}, \qquad \psi = \frac{1-\sqrt{5}}{2}.$$

These are also the two eigenvalues of Q. With

$$v_\varphi = \begin{pmatrix}\varphi\\1\end{pmatrix}, \qquad v_\psi = \begin{pmatrix}\psi\\1\end{pmatrix}, \qquad P = \begin{pmatrix}\varphi & \psi\\1 & 1\end{pmatrix},$$

one has

$$Qv_\varphi = \varphi v_\varphi, \qquad Qv_\psi = \psi v_\psi,$$

and

$$P^{\mathsf{T}}\Omega P = \sqrt{5}\begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}, \qquad P^{\mathsf{T}}GP = 5\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}.$$

The alternating form orients the ordered seam, while the symmetric form makes its endpoints null and retains their cross-pairing.

3.2 Two reflections and one phase

Normalize

$$H := \frac{G}{\sqrt{5}}, \qquad S := \frac{2Q - I}{\sqrt{5}}.$$

Proposition 3.1 (Reflection-order phase)**.** *The matrices H and S are Euclidean reflections:*

$$H^2 = S^2 = I, \qquad H^{\mathsf{T}}H = S^{\mathsf{T}}S = I, \qquad \det H = \det S = -1.$$

Their ordered products are

$$\boxed{SH = \Omega}, \qquad \boxed{HS = \Omega^{-1} = -\Omega}.$$

Consequently

$$H\Omega H = \Omega^{-1}, \qquad \langle \Omega, H\rangle \cong D_4.$$

Proof. The identities

$$G^2 = 5I, \qquad (2Q - I)^2 = 5I, \qquad 2Q - I = \Omega G, \qquad G\Omega = -\Omega G$$

follow by direct multiplication and imply every displayed relation. □

The phase is therefore generated by order: the forward product of the two reflections is the quarter-turn, while the reversed product is its inverse.

3.3 Positive phase geometry

Define

$$J := QH, \qquad B := -\Omega J, \qquad b(u,v) := u^{\mathsf{T}} B v.$$

Proposition 3.2 (Compatible complex structure and positive metric)**.** *The composite J satisfies*

$$J^2 = -I.$$

The matrix B is symmetric and positive definite, with

$$B = \frac{\sqrt{5}I - S}{2}, \qquad \det B = 1, \qquad \operatorname{tr} B = \sqrt{5}, \qquad \operatorname{spec}(B) = \{\varphi^{-1}, \varphi\}.$$

Moreover,

$$J^{\mathsf{T}} B J = B, \qquad J^{\mathsf{T}} \Omega J = \Omega, \qquad \omega(u,v) = b(u, Jv).$$

Hence

$$U_J(\theta) := e^{\theta J}$$

preserves both b and ω. In the golden basis,

$$P^{-1} J P = \Omega, \qquad P^{\mathsf{T}} B P = \sqrt{5} I.$$

Proof. Since $Q = (I + \sqrt{5}S)/2$ and $SH = \Omega$,

$$J = QH = \frac{H + \sqrt{5}\Omega}{2}.$$

Because $H^2 = I$, $\Omega^2 = -I$, and $H\Omega + \Omega H = 0$,

$$J^2 = \frac{I + 5(-I)}{4} = -I.$$

Also $\Omega H = S$, so

$$B = -\Omega J = \frac{\sqrt{5}I - S}{2}.$$

The reflection S has eigenvalues ± 1; therefore the eigenvalues of B are $(\sqrt{5}-1)/2 = \varphi^{-1}$ and $(\sqrt{5}+1)/2 = \varphi$, proving positivity and the determinant and trace claims. Direct multiplication gives $J^{\mathsf{T}} B J = B$ and $J^{\mathsf{T}} \Omega J = \Omega$. Finally $BJ = \Omega$, so $\omega(u,v) = b(u, Jv)$. The one-parameter invariance follows by differentiating $U_J(\theta)^{\mathsf{T}} B U_J(\theta)$ and $U_J(\theta)^{\mathsf{T}} \Omega U_J(\theta)$. The golden-basis formulas are direct substitutions. □

The same two golden directions therefore have three exact readings:

Ω	:	nonzero oriented separation,
G	:	two null directions of a signature-$(1,1)$ form,
B	:	orthogonal equal-norm axes for the J-phase.

No form is obtained by silently retyping another.

4 Finite phase and infinite transport

The same mirror also reverses Fibonacci transport:

$$HQH = -Q^{-1}.$$

Passing to projective classes $q = [Q]$ and $h = [H]$ removes the scalar sign.

Proposition 4.1 (Infinite-dihedral lift)**.** *In* $\mathrm{PGL}(2,\mathbb{R})$,

$$h^2 = 1, \qquad hqh = q^{-1}.$$

The element q has infinite order, and therefore

$$\langle q, h \mid h^2 = 1,\ hqh = q^{-1} \rangle \cong D_\infty.$$

The assignment

$$\rho_{\mathrm{ph}}(q) = \Omega, \qquad \rho_{\mathrm{ph}}(h) = H$$

defines a surjection

$$D_\infty \twoheadrightarrow D_4.$$

Proof. The first relation follows from $H^2 = I$, and the second from $HQH = -Q^{-1}$. The eigenvalue ratio of Q is $\varphi/\psi = -\varphi^2$, which is not a root of unity, so the projective class q has infinite order. This is the standard infinite-dihedral presentation [1]. The phase assignment respects the same inverse-conjugation relation because $H\Omega H = \Omega^{-1}$; it is surjective since Ω and H generate D_4. □

Thus the finite phase can return after four turns while the projective transport remains infinitely addressed. Quotient closure is exact; lifted state identity is not inferred.

5 An explicit Lorentzian gauge witness

Let

$$M = \mathbb{R}_t \times S^1_\vartheta \times \mathbb{R}_x \times \mathbb{R}_y,$$

and let η be the standard closed one-form on S^1, normalized by

$$d\eta = 0, \qquad \int_{S^1} \eta = 2\pi.$$

Fix $c, \rho > 0$ and define the Lorentzian metric

$$g_L = -c^2 dt^2 + \rho^2 \eta^2 + dx^2 + dy^2.$$

Take the trivial oriented rank-two bundle $E = M \times V$. The same matrix Ω now occupies a separate office as the Lie-algebra generator of $SO(2)$ in the standard Abelian connection formalism [2, 4]. Let $f \in C^2(\mathbb{R})$ with $f' \not\equiv 0$, put

$$u = t - \frac{x}{c},$$

and define the scalar connection form and matrix-valued connection

$$\mathcal{A}_a = a\eta + f(u)\, dy, \qquad \widehat{\mathcal{A}}_a = \mathcal{A}_a \Omega, \qquad a \in \mathbb{R}.$$

Its scalar curvature is

$$F = d\mathcal{A}_a = f'(u)\, du \wedge dy,$$

which is independent of a. Since $dF = 0$, it remains to check the second source-free Maxwell equation. The only nonzero coordinate components are

$$F_{ty} = f'(u), \qquad F_{xy} = -\frac{1}{c} f'(u).$$

The metric coefficients are constant, and therefore

$$\nabla_\mu F^{\mu y} = \partial_t F^{ty} + \partial_x F^{xy} = -\frac{1}{c^2} f''(u) + \frac{1}{c^2} f''(u) = 0.$$

All remaining divergences vanish. Hence, in the differential-form description of source-free electromagnetism [5],

$$\boxed{dF = 0, \qquad d \star_{g_L} F = 0, \qquad F \neq 0.}$$

The profile $u = t - x/c$ is a propagating source-free plane wave.

For a piecewise-smooth path γ, parallel transport is

$$P_a(\gamma) = \exp\left(-\Omega \int_\gamma \mathcal{A}_a\right).$$

If ℓ winds once around S^1 and has $dy = 0$, then

$$\mathrm{Hol}_a(\ell) = e^{-2\pi a\Omega}.$$

As a varies, this covers all of $SO(2)$ while F remains fixed. Thus curvature and loop holonomy are independently tunable within the explicit family.

Now fix

$$p = (0, [0], 0, 0), \qquad q = (T, [0], 0, 0), \qquad T > \frac{2\pi\rho}{c},$$

and define, for $0 \le t \le T$,

$$\gamma_0(t) = (t, [0], 0, 0), \qquad \gamma_1(t) = \left(t, \left[\frac{2\pi t}{T}\right], 0, 0\right).$$

Both paths are timelike and join p to q. Their proper times are

$$\tau[\gamma_0] = T, \qquad \tau[\gamma_1] = T\sqrt{1 - \left(\frac{2\pi\rho}{cT}\right)^2},$$

so

$$\boxed{\Delta\tau := \tau[\gamma_0] - \tau[\gamma_1] > 0.}$$

The loop $\gamma_1 * \overline{\gamma_0}$ winds once around S^1 and has $dy = 0$, so it simultaneously carries the holonomy $e^{-2\pi a\Omega}$. The proper-time difference is independent of a and of the Maxwell profile f.

Proof of Theorem 1.1. Propositions 2.1 and 2.2 establish the finite kernel and dual-form transport. Section 3 constructs the alternating, Lorentzian, and positive phase geometries and proves their exact relations. Proposition 4.1 gives the infinite-dihedral lift and finite phase quotient. The explicit data of this section satisfy the source-free Maxwell equations, fixed-curvature variable-holonomy law, and unequal-proper-time condition. The displayed carriers and codomains keep the resulting offices distinct. □

6 Strict boundary and status ledger

The following cuts are part of the result.

(a) The alternating form ω, the indefinite form g_G, and the positive form b are different tensor fields on the same vector space. Common carrier does not imply common type.

(b) G has signature $(1, 1)$, while B is positive definite. The golden directions are null for G and nonzero, orthogonal axes for B.

(c) Q reverses ω and g_G. It is not asserted to be unitary for B or for the standard Euclidean metric.

(d) D_∞ and D_4 are related by a quotient representation. They are not the same group, and phase return does not imply projective transport return.

(e) The fiber forms G and B are not the spacetime metric g_L.

(f) The Maxwell curvature $F \in \Omega^2(M)$, the holonomy $\mathrm{Hol}_a(\ell) \in SO(2)$, and the proper-time difference $\Delta\tau \in \mathbb{R}$ have different codomains and are not equated.

(g) Fixed curvature with variable holonomy is a topological connection statement on the chosen non-simply-connected manifold. It is not a claim that local electromagnetic fields determine no observable physics.

(h) The construction proves exact coexistence and independent tuning in one explicit model. It does not claim a new fundamental interaction or a derivation of spacetime from the two-dimensional carrier.

Status ledger

Office	Status	Witness
Complete local matrix algebra	CERTIFIED	Proposition 2.1
Dual-form reversal and two-step return	CERTIFIED	Proposition 2.2
Golden Lorentzian null rays	CERTIFIED	Section 3
Two reflections and ordered phase	CERTIFIED	Proposition 3.1
Compatible complex structure and positive metric	CERTIFIED	Proposition 3.2
Infinite-dihedral transport and D_4 quotient	CERTIFIED	Proposition 4.1
Nonzero source-free Maxwell curvature	CERTIFIED	Section 5
Fixed curvature with arbitrary $SO(2)$ holonomy	CERTIFIED	Section 5
Unequal proper time on shared endpoints	CERTIFIED	Section 5
Identification of fiber and spacetime metrics	EXCLUDED	Different carriers and tensor offices
Derivation of one physical quantity from another	EXCLUDED	Coexistence and independence only

Mechanical audit

The companion script `check_one_carrier_three_geometries.py` verifies the complete finite matrix algebra, pullback signs, null polynomial, reflection identities, complex-structure law, positive-metric invariants, golden-basis normal form, and the coordinate cancellation in the Maxwell divergence. These checks are regression gates for the displayed identities, not substitutes for the proofs.

7 Minimum compression

The complete carrier architecture compresses to

$$G = [Q, \Omega], \qquad Q^{\mathsf{T}}\Omega Q = -\Omega, \qquad Q^{\mathsf{T}} G Q = -G,$$
$$H = \frac{G}{\sqrt{5}}, \qquad S = \frac{2Q - I}{\sqrt{5}}, \qquad SH = \Omega, \qquad HS = \Omega^{-1},$$
$$J = QH, \qquad J^2 = -I, \qquad B = -\Omega J > 0, \qquad \operatorname{spec}(B) = \{\varphi^{-1}, \varphi\},$$
$$\langle [Q], [H] \rangle \cong D_\infty, \qquad D_\infty \twoheadrightarrow D_4,$$
$$F = d\mathcal{A}_a \neq 0, \qquad dF = d \star F = 0, \qquad \mathrm{Hol}_a(\ell) = e^{-2\pi a \Omega}, \qquad \Delta\tau > 0.$$

One carrier preserves orientation, null structure, and positive phase without requiring any two of them to be the same object.

Compatibility is proved by construction; non-coercion is part of the theorem.

Author declarations

Conflict of interest. The author declares no conflict of interest.
Author contributions. Anthony Vito Coppa: conceptualization, architecture, formal analysis, investigation, methodology, and writing.
Computational assistance. Generative language-model tools assisted with formal consolidation, typesetting, and algebraic regression checks. Every definition, proof, boundary, and conclusion was reviewed under the author's architecture.
Data availability. All mathematical data supporting the result are contained in the article and its companion regression script.

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